Variational Multiscale Method for solving Navier-Stokes equations

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Basic facts

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega\times(0,T)$$
$$\nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega\times(0,T)$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \text{ for } \mathbf{x}\in\Omega$$

$u = 0$ on $\partial\Omega$

Navier-Stokes equations:
u - velocity
p - pressure

Normalization condition: $\int_\Omega p(\mathbf{x},t)\,d\mathbf{x} = 0$

Basic physical quantatities:

kinetic energy $k(t) = \frac{1}{2}\|\mathbf{u}(t)\|^2$,

energy dissipation rate $\varepsilon(t) = \frac{\nu}{|\Omega|}\|\nabla\mathbf{u}(t)\|^2$,

power input $P(t) = (\mathbf{f}(t),\mathbf{u}(t))$,

$\nu = \frac{\mu}{\rho}$ kinematic viscosity

$Re := \frac{UL}{\nu}$, U = characteristic velocity, L = characteristic length,

ν = kinematic viscosity.

Basic facts

Non-dimensional form of incompressible Navier-Stokes equations:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{Re} \Delta \mathbf{u} + \nabla p = \mathrm{f} \quad \text{in } \Omega \times (0,T),$$
$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0,T).$$

From Kolmogorov theory it is known that small scales exist to O($Re^{-3/4}$).
This means that the mesh size should be of this order i.e. $h \sim Re^{-3/4}$
Hence the number of mesh points needed to solve directly
(DNS - Direct Navier-Stokes) is of the order $N \sim Re^{-9/4}$ in 3D.
Below some examples of Reynolds numbers:

- model airplane (characteristic length 1 m, characteristic velocity 1 m/s)
  $Re \approx 7 \cdot 10^4$
  requiring $N \approx 8 \cdot 10^{10}$ mesh points per time-step for a DNS
- cars (characteristic velocity 3 m/s)
  $Re \approx 6 \cdot 10^5$
  requiring $N \approx 10^{13}$ mesh points per time-step for a DNS
- airplanes (characteristic velocity 30 m/s)
  $Re \approx 2 \cdot 10^7$
  requiring $N \approx 2 \cdot 10^{16}$ mesh points per time-step for a DNS
- atmospheric flows
  $Re \approx 10^{20}$
  requiring $N \approx 10^{45}$ mesh points per time-step for a DNS

Basic mathematical facts

Definition (Leray–Hopf weak solutions). *We say that a measurable function* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *is a weak solution to the NSE*

1. $\mathbf{u} \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1_{0,\sigma})$;
2. *for each* $\phi \in C_0^\infty(\Omega \times [0,T))$,

   with $\nabla \cdot \phi = 0$, *the following identity holds:*

$$\int_0^\infty \int_\Omega \left[\mathbf{u}\, \phi_t - \frac{1}{Re} \nabla \mathbf{u} \nabla \phi - \mathbf{u} \cdot \nabla \mathbf{u}\, \phi \right] d\mathbf{x}\, dt$$

$$= - \int_0^\infty \int_\Omega \mathbf{f}\, \phi\, d\mathbf{x}\, dt - \int_\Omega \mathbf{u}_0\, \phi(0)\, d\mathbf{x};$$

3. *the "energy inequality" is satisfied for* $t \in [0,T]$:

$$\frac{1}{2} \|\mathbf{u}(t)\|^2 + \frac{1}{Re} \int_0^t \|\nabla \mathbf{u}(\tau)\|^2 \, d\tau \le \frac{1}{2} \|\mathbf{u}_0\|^2 + \int_0^t \int_\Omega \mathbf{f}(\mathbf{x},\tau) \mathbf{u}(\mathbf{x},\tau)\, d\mathbf{x}\, d\tau.$$

H^1 is the Sobolev space of square integrable functions with weak derivatives of order 1

$$H^1_{0,\sigma} := \{ \mathbf{u} \in [H^1_0(\Omega)]^d : \ \nabla \cdot \mathbf{u} = 0 \} \qquad \|f\|_{L^p(0,T;X)} = \begin{cases} \left[\int_0^T \|f(\tau)\|_X^p \, d\tau \right]^{1/p} & \text{if } 1 \le p < +\infty \\ \operatorname*{ess\,sup}_{0<\tau<T} \|f(\tau)\|_X & \text{if } p = +\infty \end{cases}$$

$$||u||_{H_0^1} = ||u||_{L^2} + ||\nabla u||_{L^2} \sim ||\nabla u||_{L^2}$$

Basic mathematical facts

Theorem (Leray–Hopf)
Consider $\mathbf{u}_0$ *and* $\mathbf{f}$ *with*

$$\mathbf{u}_0 \in L^2_\sigma \quad \textit{and} \quad \mathbf{f} \in L^2(0,T;L^2_\sigma).$$

Then, there exists at least one weak solution to the NSE on $[0,T]$. *Weak solutions satisfy the energy inequality that, in a bounded domain, can be rewritten in a dimensional form as*

$$k(t) + |\Omega| \int_0^t \epsilon(t')\,dt' \leq k(0) + \int_0^t P(t')\,dt', \quad \forall\, t \in [0,T].$$

It is known that weak solutions satisfy:

$$\mathbf{u}_t \in \begin{cases} L^{4/3}(0,T;(H^1_{0,\sigma})') & \text{if } \Omega \subset \mathbb{R}^3 \\ L^2(0,T;(H^1_{0,\sigma})') & \text{if } \Omega \subset \mathbb{R}^2. \end{cases}$$

Definition *We say that a weak solution* $\mathbf{u}$ *is a strong solution if*

$$\begin{cases} \mathbf{u} \in L^\infty(0,T;H^1_{0,\sigma}) \cap L^2(0,T;H^1_{0,\sigma} \cap [H^2(\Omega)]^d), \\ \mathbf{u}_t \in L^2(0,T;L^2_\sigma), \end{cases}$$

where $H^2(\Omega) \subset L^2(\Omega)$ *is the space of (classes of equivalence of) functions in* $L^2(\Omega)$ *with derivatives up to the second order in* $L^2(\Omega)$.

Basic mathematical facts

1. Strong solutions are unique also in a wider class of weak solutions, but it is not known whether they exist.

Theorem *Let* $\mathbf{u}_0 \in H^1_{0,\sigma}$ *and* $\mathbf{f} \in L^2(0,T;L^2_\sigma)$. *Then there exists* $0 < T_0 \le T$ *such that there exists a unique strong solution in* $[0,T_0)$. *The time* T_0 *depends on* $\mathbf{f}, \|\nabla \mathbf{u}_0\|$, *and* Re;

2. Strong solutions satisfy energy equality
3. Strong solution become smooth (for each positive time) if ∂Ω, u_o and f are smooth.

Theorem *Let* $\mathbf{u}$ *be a strong solution in* $[0,T]$. *If* Ω *is of class* C^∞ *and if* $\mathbf{f} \in C^\infty((0,T] \times \overline{\Omega})$ *then*

$$\mathbf{u} \in C^\infty([\varepsilon,T] \times \overline{\Omega}), \quad \forall \varepsilon > 0.$$

In fact to have smooth solutions it is sufficient to know that:

$$\mathbf{u} \in L^r(0,T;L^s(\Omega)) \quad \text{for} \quad \frac{2}{r} + \frac{d}{s} = 1.$$

So, for d=2 r=2, s=2 – which is true for weak solutions.

Technically most of the proofs use the Ladyzhenskaya inequality:

$$\|u\|_{L^4} \le \begin{cases} 2^{1/4}\|u\|^{1/2}\|\nabla u\|^{1/2} & \textit{if } \Omega \subset \mathbb{R}^2, \\ 4^{1/4}\|u\|^{1/4}\|\nabla u\|^{3/4} & \textit{if } \Omega \subset \mathbb{R}^3. \end{cases}$$

Basic mathematical facts

Question: to what extend irregularity of the solutions exist?

Definition *We say that a solution* $\mathbf{u}$ *becomes irregular at the time* T^* *if and only if*

(a) $T^* < \infty$;
(b) $\mathbf{u} \in C^\infty((s,T^*) \times \overline{\Omega})$, *for some* $s < T^*$;
(c) it is not possible to extend $\mathbf{u}$ *to a regular solution in any interval* (s, T^{**}), *with* $T^{**} > T^*$.

The number T^* is called the *epoch of irregularity* ("époque de irrégularité" in Leray).

Theorem (Leray, Scheffer). *Let* $\mathbf{u}$ *be a weak solution and let* T^* *be an epoch of irregularity. Then the following properties hold:*

1. $\|\nabla \mathbf{u}(t)\| \to \infty$ *as* $t \to T^*$ *in such a way that,*

$$\exists\, C = C(\Omega) > 0 : \qquad \|\nabla \mathbf{u}(t)\| \leq \frac{C}{Re^{3/4}(T^* - t)}, \qquad \forall\, t < T^*;$$

2. *the 1/2-dimensional Hausdorff dimension of the set of (possible) epochs of irregularity is equal to zero.*

This means that the set of irregular solutions of NS equations in 3D is fractal !

Some notations

Time averages:

$$\langle \mathbf{u} \rangle(\mathbf{x}) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{u}(\mathbf{x},t)\,dt, \qquad \langle p \rangle(\mathbf{x}) := \lim_{T\to\infty} \frac{1}{T}\int_0^T p(\mathbf{x},t)\,dt.$$

Space averages:

The simplest form: $$\overline{\mathbf{u}}(\mathbf{x},t) = \frac{1}{\delta^3}\int_{x_1-\frac{\delta}{2}}^{x_1+\frac{\delta}{2}}\int_{x_2-\frac{\delta}{2}}^{x_2+\frac{\delta}{2}}\int_{x_3-\frac{\delta}{2}}^{x_3+\frac{\delta}{2}} \mathbf{u}(y_1,y_2,y_3,t)\,dy_1dy_2dy_3$$

has many disadvantages, hence approach based on convolution is used:

$$0 \le g(\mathbf{x}) \le 1, \qquad g(\mathbf{0}) = 1, \qquad \int_{\mathbb{R}^d} g(\mathbf{x})\,d\mathbf{x} = 1. \qquad g_\delta(\mathbf{x}) := \frac{1}{\delta^d} g\left(\frac{\mathbf{x}}{\delta}\right)$$

$$\overline{\mathbf{u}}(\mathbf{x},t) = (g_\delta * \mathbf{u})(\mathbf{x},t) := \int_{\mathbb{R}^d} g_\delta(\mathbf{x}-\mathbf{x}')\mathbf{u}(\mathbf{x}',t)\,d\mathbf{x}', \quad \text{and } \mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}.$$

Example:
Gaussian filter

$$g(\mathbf{x}) := \left(\frac{\gamma}{\pi}\right)^{3/2} \frac{1}{\delta^3} \mathrm{e}^{-\dfrac{\gamma\,|\,\mathbf{x}\,|^2}{\delta^2}}$$

$$\widehat{g}(\mathbf{k}) = \mathrm{e}^{-\dfrac{\delta^2\,|\,\mathbf{k}\,|^2}{4\gamma}}$$

Conventional turbulence models

Time average Navier Stokes equations:

$$-\frac{1}{Re}\triangle\langle \mathbf{u}\rangle + \nabla\cdot\langle \mathbf{u}\,\mathbf{u}\rangle + \nabla\langle p\rangle = \langle \mathbf{f}\rangle, \quad \text{and} \quad \nabla\cdot\langle \mathbf{u}\rangle = 0, \quad \text{in } \Omega.$$

Since u = <u> + u' this leads to:

$$-\frac{1}{Re}\triangle\langle \mathbf{u}\rangle + \nabla\cdot\langle \mathbf{u}\rangle\langle \mathbf{u}\rangle + \nabla\cdot\langle \mathbf{u}'\mathbf{u}'\rangle + \nabla\langle p\rangle = \langle \mathbf{f}\rangle, \text{ and } \nabla\cdot\langle \mathbf{u}\rangle = \mathbf{0}, \text{ in } \Omega.$$

Due to the fact that $\langle \mathbf{u}\,\mathbf{u}\rangle \neq \langle \mathbf{u}\rangle\langle \mathbf{u}\rangle$ some model is needed for ∇·<u'u'>

Example:

$$\nabla\cdot\langle \mathbf{u}'\,\mathbf{u}'\rangle \approx -\nabla\cdot(\nu_T\nabla^s\langle \mathbf{u}\rangle) + \text{ terms incorporated into the pressure.}$$

$$\nu_T = Constant\ l\ \langle\sqrt{k'}\rangle, \quad (\nabla^s \mathbf{v})_{ij} := \frac{1}{2}(v_{i,x_j} + v_{j,x_i})$$

$$l = l(\mathrm{x}, t): \text{ local length scale of turbulent fluctuations,}$$

$$k' = \frac{1}{2}|\mathbf{u}'(\mathrm{x}, t)|^2: \text{ kinetic energy of turbulent fluctuations.}$$

This is linked to RANS – Reynolds Average Navier-Stokes

LES: Large Eddy Simulations

$$\overline{\mathbf{u}}(\mathbf{x},t) := \int_{\mathbb{R}^d} \mathbf{u}(\mathbf{x}-\mathbf{x}',t)\, g_\delta(\mathbf{x}')\, d\mathbf{x}' \qquad \overline{p}(\mathbf{x},t) := \int_{\mathbb{R}^d} p(\mathbf{x}-\mathbf{x}',t)\, g_\delta(\mathbf{x}')\, d\mathbf{x}'$$

$$\begin{aligned} \overline{\mathbf{u}}_t + \nabla\cdot(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^T) - \frac{1}{Re}\Delta\overline{\mathbf{u}} + \nabla\overline{p} + \nabla\cdot(\overline{\mathbf{u}\,\mathbf{u}^T} - \overline{\mathbf{u}}\,\overline{\mathbf{u}}^T) &= \overline{\mathbf{f}} + A_\delta(\mathbf{u},p) \\ \nabla\cdot\overline{\mathbf{u}} &= 0. \end{aligned}$$

$$\overline{\mathbf{u}}\cdot\mathbf{n} = 0 \quad \text{and} \quad \beta\,\overline{\mathbf{u}}\cdot\boldsymbol{\tau}_j - \mathbf{n}\cdot\boldsymbol{\sigma}(\overline{\mathbf{u}},\overline{p})\cdot\boldsymbol{\tau}_j = 0 \qquad \text{on } \partial\Omega,$$

where boundary commutator error (BCE) term is: $A_\delta(\mathbf{u},p) = \int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\,\boldsymbol{\sigma}(\mathbf{u},p)(\mathbf{s})\cdot\mathbf{n}(\mathbf{s})\,dS(\mathbf{s})$

Subfilter-scale stress tensor $\tau = \overline{\mathbf{u}\,\mathbf{u}} - \overline{\mathbf{u}}\,\overline{\mathbf{u}} \approx \mathcal{S}(\overline{\mathbf{u}},\overline{\mathbf{u}})$.

Total stress: $\boldsymbol{\sigma}(\overline{\mathbf{u}},\overline{p}) := \overline{p}\,\mathbb{I} - \frac{2}{Re}\nabla^s\overline{\mathbf{u}} + \mathcal{S}(\overline{\mathbf{u}},\overline{\mathbf{u}})$.

$$\begin{aligned} &\overline{\mathbf{u}}_t - \frac{2}{Re}\nabla\cdot(\nabla^s\overline{\mathbf{u}}) + \nabla\cdot(\overline{\mathbf{u}\mathbf{u}^T}) + \nabla\overline{p} = \mathbf{f} \\ &\quad + \int_{\partial\Omega} g(\mathbf{x}-\mathbf{s})\left[\frac{2}{Re}\nabla^s\mathbf{u}(\mathbf{s})\cdot\mathbf{n}(\mathbf{s}) - p(\mathbf{s})\mathbf{n}(\mathbf{s})\right] dS(\mathbf{s}) \quad \text{in } (0,T)\times\mathbb{R}^d. \end{aligned}$$

BCE can be estimated as follows:

$$\int_{\mathbb{R}^d}\left|\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})\,dS(\mathbf{s})\right|^k d\mathbf{x} \le C\delta^{1+k\left(\frac{(d-1)\alpha}{q}-d\right)}\|\psi\|^k_{L^p(\partial\Omega)}$$

LES: Large Eddy Simulations

Finally it leads to the following formulation:

$$\mathrm{w}_t + \nabla \cdot (\mathrm{w}\,\mathrm{w}^T) - \nabla \cdot \left(\frac{2}{Re} \nabla^s \mathrm{w} - \mathcal{S}(\mathrm{w},\mathrm{w}) \right) + \nabla q = \overline{\mathrm{f}} \text{ in } \Omega \times (0,T]$$

$$\nabla \cdot \mathrm{w} = 0 \quad \text{in } \Omega \times (0,T]$$

$$\mathrm{w}(\mathrm{x},0) = \overline{\mathrm{u}}_0(\mathrm{x}) \quad \text{in } \Omega$$

$$\overline{\mathrm{w}} \cdot \mathrm{n} = 0 \quad \text{and} \quad \beta\, \overline{\mathrm{w}} \cdot \boldsymbol{\tau}_j - \mathrm{n} \cdot \boldsymbol{\sigma}(\overline{\mathrm{w}}, \overline{p}) \cdot \boldsymbol{\tau}_j = 0 \text{ on } \partial\Omega \times (0,T]$$

$$\boldsymbol{\sigma}(\mathrm{w},q) := q\,\mathbb{I} - \tfrac{2}{Re} \nabla^s \mathrm{w} + \mathcal{S}(\mathrm{w},\mathrm{w})$$

Variational formulation of LES:

$$(\mathrm{w}_t, \mathrm{v}) + (\mathrm{w} \cdot \nabla \mathrm{w}, \mathrm{v}) + \left(\frac{2}{Re} \nabla^s \mathrm{w} - \mathcal{S}(\mathrm{w},\mathrm{w}), \nabla \mathrm{v} \right) + \Gamma - (q, \nabla \cdot \mathrm{v}) = (\overline{\mathrm{f}}, \mathrm{v})$$

$$\Gamma = -\int_{\partial\Omega} \mathrm{n} \cdot \left(\frac{2}{Re} \nabla^s \mathrm{w} - \mathcal{S}(\mathrm{w},\mathrm{w}) \right) \cdot \mathrm{v}\, dS$$

Decomposition of v: $\mathrm{v} = (\mathrm{v} \cdot \mathrm{n})\,\mathrm{n} + (\mathrm{v} \cdot \boldsymbol{\tau}_j)\,\boldsymbol{\tau}_j = (\mathrm{v} \cdot \boldsymbol{\tau}_j)\,\boldsymbol{\tau}_j$ as (v,n)=0.

leads to: $\Gamma = \int_{\partial\Omega} \beta(\mathrm{w})\, \mathrm{w} \cdot \boldsymbol{\tau}_j\, \mathrm{v} \cdot \boldsymbol{\tau}_j\, dS$

LES: variational formulation

$$\mathbf{X} := \left\{ \mathrm{v} \in [H^1(\Omega)]^d : \ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\}$$

$$Q := \left\{ q \in L^2(\Omega) : \ \int_\Omega q \, d\mathrm{x} = 0 \right\},$$

Find velocity *w*:[0,T]→X, and pressure *q* such that for any *v*:

$$\begin{cases} (\mathbf{w}_t, \mathbf{v}) + (\mathbf{w} \cdot \nabla \mathbf{w}, \mathbf{v}) + \left(\dfrac{2}{Re} \nabla^s \mathbf{w} - \mathcal{S}(\mathbf{w}, \mathbf{w}), \nabla^s \mathbf{v} \right) \\ \qquad + \displaystyle\int_{\partial\Omega} \beta(\mathbf{w}) \, \mathbf{w} \cdot \boldsymbol{\tau}_j \, \mathbf{v} \cdot \boldsymbol{\tau}_j \, dS - (q, \nabla \cdot \mathbf{v}) = (\overline{\mathbf{f}}, \mathbf{v}), \quad \forall \, \mathbf{v} \in \mathbf{X}, \\ \qquad\qquad (\nabla \cdot \mathbf{w}, \lambda) = 0, \quad \forall \, \lambda \in Q. \end{cases}$$

This is so called *mixed* variational formulation: spaces here are not divergence-free and the constraint is imposed in an approximate way. If *S(w,w)*=0 additional analysis is required to take into account boundary estimation error. For the boundary condition *u=0* the space $(H_0^1)^d$ is used.

LES: numerical variational formulation

Let $X_h \subset X$ and $Q_h \subset Q$ are finite dimensional subspaces (for example based on finite elemet method). Then the problem is to find:

$\mathbf{w}^h : [0,T] \to \mathbf{X}_h, \qquad q^h : (0,T] \to Q_h$ such that,

$$\begin{cases} (\mathbf{w}_t^h, \mathbf{v}^h) + b^*(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) + \left(\frac{2}{Re} \nabla^s \mathbf{w}^h - \mathcal{S}(\mathbf{w}^h, \mathbf{w}^h), \nabla^s \mathbf{v}^h \right) \\ \qquad + \int_{\partial\Omega} \beta(\mathbf{w}^h)\, \mathbf{w}^h \cdot \boldsymbol{\tau}_j\, \mathbf{v}^h \cdot \boldsymbol{\tau}_j \, dS - (q^h, \nabla \cdot \mathbf{v}^h) = (\overline{\mathbf{f}}, \mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{X}_h, \\ \qquad\qquad (\nabla \cdot \mathbf{w}^h, \lambda^h) = 0, \quad \forall\, \lambda^h \in Q_h. \end{cases}$$

where $\quad b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v})$.

Fast reminder basic fact from numerical analysis (Lax theorem):
Approximation + Stability => Convergence

Stability can be achieved by adding extra stabilization term or by satisfying special conditions

Fundamentals in numerical analysis

Differential equation:

$$L: U \to F, \quad l: U \to G$$
$$Lu = f, \; u \in U, \quad f \in F$$
$$lu = g, \quad u \in U, \; g \in G$$

Approximation:

$$L_h : U_h \to F_h, \quad l_h : U_h \to G_h$$
$$L_h u_h = f_h, \; u_h \in U_h, \quad f_h \in F_h$$
$$l_h u_h = g_h, \quad u_h \in U_h, \; g_h \in G_h$$

L – differential operator, *l* – operator describing boundary/initial conditions
L_h, l_h – discrete approximation of *L* and *l*

$\{U_h, r_h^U, p_h^U\}_{h \in \omega}$ This triple is called approximation of the space U, where:

$r_h^U : U \to U_h$ r_h are restriction operators e.g.

$r_h^F : F \to F_h$ $(r_h u)(x_i) = u(x_i) \; or \; (r_h u)(x_i) = \frac{1}{\mu(B(x_i, r))} \int_{B(x_i, r)} u(x) dx$

$r_h^G : G \to G_h$

$p_h^U : U_h \to U$ p_h is prolongation operator e.g. interpolation function

Fundamentals in numerical analysis

$$L_h r_h^U u(p) - f_h(p) = O(h^q)$$
$$l_h r_h^U u(p) - g_h(p) = O(h^q)$$

Local consistency (approximation)

$$\| L_h r_h^U u - f_h \|_h^{F_h} = O(h^q)$$
$$\| l_h r_h^U u - g_h \|_h^{G_h} = O(h^q)$$

Global consistency

Approximation of space *U* is convergent if: $\pi_h^U \to I \quad \pi_h^U = p_h^U r_h^U : U \to U$

The norms are consistent if: $\forall u \in U \; \| r_h^U u \|_h \to \| u \|$

The numerical scheme is convergent if: $\| r_h^U u - u_h \|_h^{U_h} \to 0$

The numerical scheme is stable, if for any $h<h_0$ there exists unique solution of approximate equation and:

$$\| u_h \|_h^{U_h} \le M(\| f_h \|_h^{F_h} + \| g_h \|_h^{G_h})$$

Fundamentals in numerical analysis

Lax Theorem

If the numerical scheme is consistent with the order *q* (in norm sense) and stable then the scheme is convergent and:

$$\| r_h^U u - u_h \|_h^{U_h} = O(h^q)$$

$$\| u_h \|_{h,\infty} = \max_{p \in \Omega_h} | u_h(p) |$$

$$| u_h \|_{h,2} = \sqrt{h_x h_y \sum_{p \in \Omega_h} | u_h(p) |^2}$$

Cea Lemma

For finite element method:

$$\| u - u_h \| \le C \inf_{v_h \in U_h} \| u - v_h \|$$

$$u_h = \{ u(p) : p \in \Omega_h \}$$

General form of numerical scheme:

$$\sum_{q \in N_h(p)} A(p,q) u_h(q) = u_h(p), \;\; p \in \Omega_h \cup \Gamma_h$$

N_h – grid neighbourhood

Example 1. If for some α independent of h the following condition holds:

$$A(p,p) - \sum_{q \in N'_h(p)} | A(p,q) | \ge \alpha, \;\; N'_h(p) = N_h(p) - \{p\}$$

then the scheme is stable in the norm „max".

Example 2. For advection problems CFL (Courant-Friedrichs-Levy) condition for explicit methods:

$$v \frac{\Delta t}{(\Delta x)^p} \le C$$

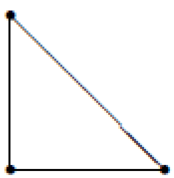
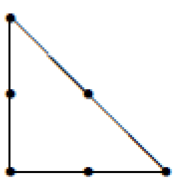
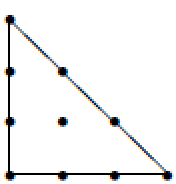
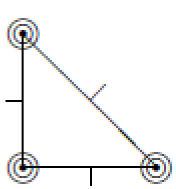
Some standard finite elements examples in 2D

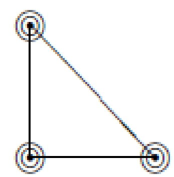
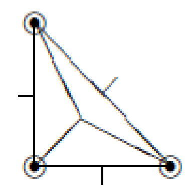
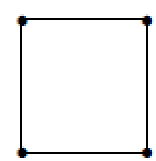
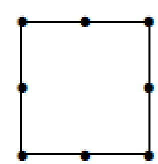
$$\mathcal{M}^k := \mathcal{M}_k(\mathcal{T}) := \{v \in L_2(\Omega);\ v|_T \in \mathcal{P}_k \text{ for every } T \in \mathcal{T}\},$$

$$\mathcal{M}_0^k := \mathcal{M}^k \cap C^0(\Omega) = \mathcal{M}^k \cap H^1(\Omega),$$

$$\mathcal{M}_{0,0}^k := \mathcal{M}^k \cap H_0^1(\Omega).$$

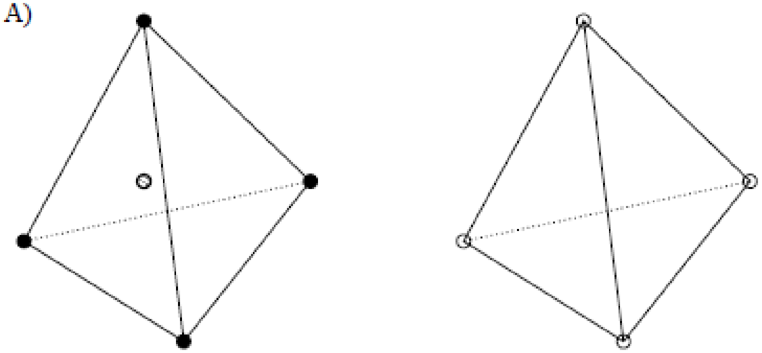
- • Function value prescribed
- ⊙ Function value and 1st derivative prescribed
- ◎ Function value and 1st and 2nd derivatives prescribed
- ⊥ Normal derivative prescribed

	Linear triangular element $\mathcal{M}_0^1$ $u \in C^0(\Omega)$ $\Pi_{\text{ref}} = \mathcal{P}_1$, $\dim \Pi_{\text{ref}} = 3$
	Quadratic triangular element $\mathcal{M}_0^2$ $u \in C^0(\Omega)$ $\Pi_{\text{ref}} = \mathcal{P}_2$, $\dim \Pi_{\text{ref}} = 6$
	Cubic triangular element $\mathcal{M}_0^3$ $u \in C^0(\Omega)$ $\Pi_{\text{ref}} = \mathcal{P}_3$, $\dim \Pi_{\text{ref}} = 10$
	Argyris triangle $u \in C^1(\Omega)$ $\Pi_{\text{ref}} = \mathcal{P}_5$, $\dim \Pi_{\text{ref}} = 21$

	Bell triangle $u \in C^1(\Omega)$ $\Pi_{\text{ref}} \subset \mathcal{P}_5$, $\partial_\nu u|_{\partial T_i} \in \mathcal{P}_3$, $\dim \Pi_{\text{ref}} = 18$
	Hsieh–Clough–Tocher element $u \in C^1(\Omega)$ $T = \bigcup_{i=1}^3 K_i$, $u|_{K_i} \in \mathcal{P}_3$, $\dim \Pi_{\text{ref}} = 12$
	Bilinear quadrilateral element Q_1 $u \in C^0(\Omega)$ $\Pi_{\text{ref}} \subset \mathcal{P}_2$, $u|_{\partial T_i} \in \mathcal{P}_1$, $\dim \Pi_{\text{ref}} = 4$
	Serendipity element $u \in C^0(\Omega)$ $\Pi_{\text{ref}} \subset \mathcal{P}_3$, $u|_{\partial T_i} \in \mathcal{P}_2$, $\dim \Pi_{\text{ref}} = 8$

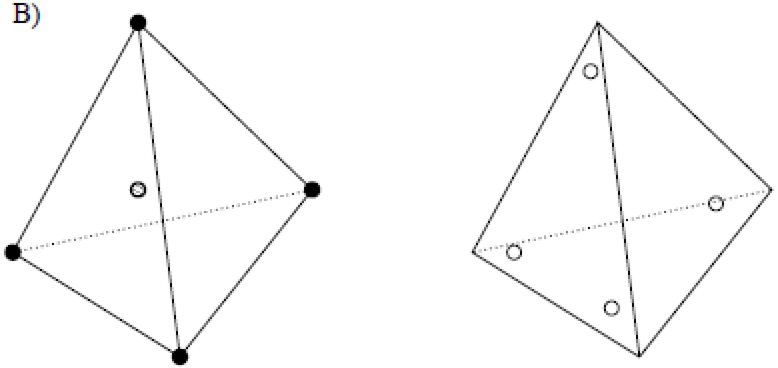
Some 3D elements

A)



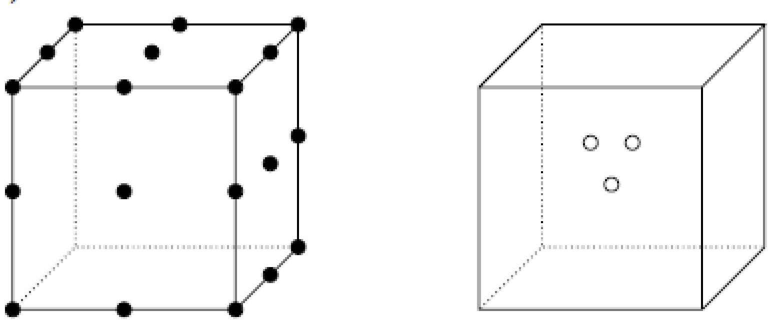
MINI element
(with bubble function)

B)



Crouseix-Raviart element
with bubble function
(non-conforming)

C)



Q_2-P_1 element

LES: stability problem

If LBB condition (Ladyzhenskaya,Babuška,Brezzi):

$$\inf_{q^h \in Q_h} \sup_{\mathbf{v}^h \in \mathbf{X}_h} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{\|q^h\| \, \|\nabla \mathbf{v}^h\|} \geq C > 0$$ is satisfied

then the following inequality holds:

$$\frac{1}{2}\|\mathbf{w}^h(t)\|^2 + \int_0^t \left[\frac{1}{Re}\|\nabla^s \mathbf{w}^h\|^2 - (\mathcal{S}(\mathbf{w}^h, \mathbf{w}^h), \nabla^s \mathbf{v}^h) + \int_{\partial\Omega} \beta(\mathbf{w}^h) \, |\mathbf{w}^h \cdot \boldsymbol{\tau}_j|^2 \, dS \right] dt' \leq \frac{1}{2}\|\overline{\mathbf{u}}_0\|^2 + C \, Re \int_0^t \|\overline{\mathbf{f}}\|_{-1}^2 \, dt'.$$

If, additionally, $\beta(\cdot) \geq \beta_0 > 0$ *and the model in dissipative in the sense that*

$$(\mathcal{S}(\mathbf{v}, \mathbf{v}), \nabla^s \mathbf{v}) \leq 0 \qquad \forall \, \mathbf{v} \in \mathbf{X},$$

then the method is stable.

Condition $\beta(\mathbf{w}) = \beta(\mathbf{w}, \delta, Re) \geq \beta_0 = \beta_0(\delta, Re) > 0.$ should be true for reasonable boundary condition, while dissipativity $\int_\Omega \mathcal{S}(\mathbf{v}, \mathbf{v}) : \nabla^s \mathbf{v} \, dx \leq 0 \qquad \forall \, \mathbf{v} \in \mathbf{X}$ holds for example for eddy viscosity models, $\mathcal{S}^*(\mathbf{v}, \mathbf{v}) = -\nu_T(\delta, \mathbf{v}) \, \nabla^s \mathbf{v}, \;\; \nu_T \geq 0$ but is not universal.

Variational Multi-scale Model

$$\mathbf{X} := \{\mathbf{v} \in [H^1(\Omega)]^d : \ \mathbf{v}|_{\partial\Omega} = 0\},$$

$$Q := \left\{ q \in L^2(\Omega) : \ \int_\Omega q \, d\mathbf{x} = (q, 1) = 0 \right\}$$

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : \ (\nabla \cdot \mathbf{v}, q) = 0, \ \forall \, q \in Q\}$$

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \frac{1}{Re} \nabla^s \mathbf{u} : \nabla^s \mathbf{v} \, d\mathbf{x},$$

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2} \int_\Omega [\mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} - \mathbf{u} \cdot \nabla \mathbf{w} \cdot \mathbf{v}] \, d\mathbf{x}$$

Basic properties of the form *b*:

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \text{and} \quad b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad \forall \, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}.$$

Problem:

Find u : [0, *T*] → X and *p* : (0, *T*] → *Q* satisfying:

$$\begin{cases} (\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}) & \forall \, \mathbf{v} \in \mathbf{X}, \\ (q, \nabla \cdot \mathbf{u}) = 0 & \forall \, q \in Q, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) & \forall \, \mathbf{x} \in \Omega. \end{cases}$$

Variational Multi-scale Model

Space decomposition: $\mathbf{X} = \overline{\mathbf{X}} \oplus \mathbf{X}'$, where $\overline{\mathbf{X}} := \mathbf{X}^h$ is the chosen finite element space.

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}', \qquad \overline{\mathbf{u}} = \mathbf{u}^h := \overline{P}\mathbf{u} \in \mathbf{X}^h, \qquad \mathbf{u}' = (\mathbb{I} - \overline{P})\,\mathbf{u} \in \mathbf{X}',$$

where $\overline{P} : \mathbf{X} \to \overline{\mathbf{X}} = \mathbf{X}^h$ is the projection operator.

Insert u = u^h + u' and alternately:
v = v^h then v = v' which gives two coupled equations:

$$\begin{cases} (\overline{\mathbf{u}}_t + \mathbf{u}'_t, \mathbf{v}^h) + a(\overline{\mathbf{u}} + \mathbf{u}', \mathbf{v}^h) + b(\overline{\mathbf{u}} + \mathbf{u}', \overline{\mathbf{u}} + \mathbf{u}', \mathbf{v}^h) - (p^h + p', \nabla \cdot \mathbf{v}^h) \\ \qquad = (\mathbf{f}, \mathbf{v}^h) \qquad \forall\, \mathbf{v}^h \in \mathbf{X}^h, \\ \\ (\overline{\mathbf{u}}_t + \mathbf{u}'_t, \mathbf{v}') + a(\overline{\mathbf{u}} + \mathbf{u}', \mathbf{v}') + b(\overline{\mathbf{u}} + \mathbf{u}', \overline{\mathbf{u}} + \mathbf{u}', \mathbf{v}') - (p^h + p', \nabla \cdot \mathbf{v}') \\ \qquad = (\mathbf{f}, \mathbf{v}') \qquad \forall\, \mathbf{v}' \in \mathbf{X}'. \end{cases}$$

This system of equations is completely equivalent to the original one !

Some algebraic manipulations lead to the following formulation:

Variational Multi-scale Model

$$(\overline{\mathbf{u}}_t, \mathbf{v}^h) + a(\overline{\mathbf{u}}, \mathbf{v}^h) + b(\overline{\mathbf{u}}, \overline{\mathbf{u}}, \mathbf{v}^h) - (p^h, \nabla \cdot \mathbf{v}^h) - (\mathbf{f}^h, \mathbf{v}^h) = (\mathbf{r}', \mathbf{v}^h).$$

where:

$$\begin{aligned}(\mathbf{r}', \mathbf{v}^h) &:= (\mathbf{f}', \mathbf{v}^h) - b(\mathbf{u}', \mathbf{u}', \mathbf{v}^h)\\ &- \left[(\mathbf{u}'_t, \mathbf{v}^h) + a(\mathbf{u}', \mathbf{v}^h) + b(\overline{\mathbf{u}}, \mathbf{u}', \mathbf{v}^h) + b(\mathbf{u}', \overline{\mathbf{u}}, \mathbf{v}^h - (p', \nabla \cdot \mathbf{v}^h)\right]\end{aligned}$$

and

$$(\mathbf{u}'_t, \mathbf{v}^h) + a(\mathbf{u}', \mathbf{v}') + b(\mathbf{u}', \mathbf{u}', \mathbf{v}') - (p', \nabla \cdot \mathbf{v}') - (\mathbf{f}', \mathbf{v}') = (\mathbf{r}^h, \mathbf{v}'),$$

where:

$$\begin{aligned}(\mathbf{r}^h, \mathbf{v}') &:= (\overline{\mathbf{f}}, \mathbf{v}') - b(\overline{\mathbf{u}}, \overline{\mathbf{u}}, \mathbf{v}')\\ &- \left[(\overline{\mathbf{u}}_t, \mathbf{v}') + a(\overline{\mathbf{u}}, \mathbf{u}') + b(\mathbf{u}', \overline{\mathbf{u}}, \mathbf{v}') + b(\overline{\mathbf{u}}, \mathbf{u}', \mathbf{v}') - (\overline{p}, \nabla \cdot \mathbf{v}')\right]\end{aligned}$$

In VMM these two equations are discretized simultaneously:
for X^h chosen complementary finite dimensional X'_b is taken for fluctuation approximation.
Because of stability problem additional term is added of the form:
$(\nu_T(u)\nabla u, \nabla v)$.

Variational Multi-scale Model

Find: $\overline{\mathbf{u}} : [0,T] \to \mathbf{X}^h, \quad \overline{p} : (0,T] \to Q^h,$ $\mathbf{u}'_b : [0,T] \to \mathbf{X}'_b, \quad p' : (0,T] \to Q'_b$ such that:

$$(\overline{\mathbf{u}}_t, \mathbf{v}^h) + a(\overline{\mathbf{u}}, \mathbf{v}^h) + b(\overline{\mathbf{u}}, \overline{\mathbf{u}}, \mathbf{v}^h) - (p^h, \nabla \cdot \mathbf{v}^h) + (q^h, \nabla \cdot \overline{\mathbf{u}}) - (\mathbf{f}^h, \mathbf{v}^h)$$
$$= (\mathbf{r}'_b, \mathbf{v}^h) \quad \forall\, \mathbf{v}^h \in \mathbf{X}^h,\ q^h \in Q^h \quad (11.17)$$

where:

$$(\mathbf{r}'_b, \mathbf{v}^h) := (\mathbf{f}', \mathbf{v}^h) - b(\mathbf{u}', \mathbf{u}', \mathbf{v}^h) \ - \ [(\mathbf{u}'_{bt}, \mathbf{v}^h) + a(\mathbf{u}'_b, \mathbf{v}^h) + b(\overline{\mathbf{u}}, \mathbf{u}'_b, \mathbf{v}^h)$$
$$+\ b(\mathbf{u}'_b, \overline{\mathbf{u}}, \mathbf{v}^h) - (p'_b, \nabla \cdot \mathbf{v}^h)]$$

and

$$(\mathbf{u}'_{b,t}, \mathbf{v}'_b) + a(\mathbf{u}'_b, \mathbf{v}'_b) + (\nu_T(\overline{\mathbf{u}} + \mathbf{u}'_b)\, \nabla^s \mathbf{u}'_b, \nabla^s \mathbf{v}'_b) + b(\mathbf{u}'_b, \mathbf{u}'_b \mathbf{v}'_b)$$
$$-(p'_b, \nabla \cdot \mathbf{v}'_b) + (q'_b, \nabla \cdot \mathbf{u}'_b) = (\mathbf{r}^h, \mathbf{v}'), \quad \forall\, \mathbf{v}'_b \in \mathbf{X}'_b,\ \forall\, q'_b \in Q'_b,$$

where:

$$(\mathbf{r}^h, \mathbf{v}'_b) := (\overline{\mathbf{f}}, \mathbf{v}'_b) - b(\overline{\mathbf{u}}, \overline{\mathbf{u}}, \mathbf{v}'_b)$$
$$-\ [(\overline{\mathbf{u}}_t, \mathbf{v}'_b) + a(\overline{\mathbf{u}}, \mathbf{v}'_b) + b(\mathbf{u}'_b, \overline{\mathbf{u}}, \mathbf{v}'_b) + b(\overline{\mathbf{u}}, \mathbf{u}'_b, \mathbf{v}'_b) - (p^h, \nabla \cdot \mathbf{v}'_b)].$$

$$\nu_T = (C_s \delta)^2\, |\nabla^s (\overline{\mathbf{u}} + \mathbf{u}'_b)|, \quad \nu_T = (C_s \delta)^2\, |\nabla^s \mathbf{u}'_b|$$

with $\mathbf{u}^h(0) = \overline{\mathbf{u}}_0 = \overline{P}\mathbf{u}_0 \text{ in } \Omega, \qquad \mathbf{u}'_b(0) = \mathbf{u}'_0 \approx (\mathbb{I} - \overline{P})\, \mathbf{u}_0 \text{ in } \Omega,$

VMM typically uses a computational model for the fluctuations that uncouples second equation into one small system per mesh cell – for example using bubble functions: $\phi_{Kh} > 0$ on Kh and $\phi_{Kh} = 0$ on ∂Kh (Kh – finite elements), then

$$\mathbf{X}'_b := \operatorname{span}\{\phi_{K^h} :\ \text{all mesh cells } K^h\}^3.$$

Variational Multi-scale Model

The following theorem shows stability of VMM:

Let $\nu_T \geq 0$. $\widetilde{\mathbf{X}} := \mathbf{X}^h \oplus \mathbf{X}'_b$ *and let* $\tilde{\mathbf{u}} = \overline{\mathbf{u}} + \mathbf{u}'_b$. *Then* $\tilde{\mathbf{u}}$ *satisfies:*

$$\frac{1}{2}\|\tilde{\mathbf{u}}(t)\|^2 + \int_0^t \left(\frac{1}{Re}\|\nabla^s \tilde{\mathbf{u}}(t')\|^2 + \int_\Omega \nu_T(\mathbf{u})|\nabla^s \mathbf{u}'(t')|^2 \, d\mathbf{x} \right) dt'$$
$$= \frac{1}{2}\|\mathbf{u}_0\|^2 + \int_0^t (\mathbf{f}(t'), \tilde{\mathbf{u}}(t'))\, dt'.$$

Multiscale approach: let $\pi^H(\Omega)$ denotes coarse finite element mesh and $\pi^h(\Omega)$ finer mesh (h<H) that can be obtained by refining. Then:

$$Q^H \subset Q^h \subset Q := L_0^2(\Omega), \text{ and } \mathbf{X}^H \subset \mathbf{X}^h \subset \mathbf{X} := [H_0^1(\Omega)]^3.$$

Assume LBB condition is satisfied: $\inf_{q^\mu \in Q^\mu} \sup_{\mathbf{v}^\mu \in \mathbf{X}^\mu} \frac{(q^\mu, \nabla \cdot \mathbf{v}^\mu)}{\|q^\mu\| \, \|\nabla \mathbf{v}^\mu\|} \geq \beta > 0, \quad \text{for } \mu = h, H.$

The key is to construct multiscale decomposition of deformation tensor $\nabla^s u^h$ since $u^h \in X$ naturally we have:

$$\nabla^s \mathbf{u}^h \in \mathbf{L} := \{\ell = \ell_{ij} : \ell_{ij} = \ell_{ji} \text{ and } \ell_{ij} \in L^2(\Omega), \; i, j = 1, 2, 3.\}$$

Variational Multi-scale Model

Then discontinouous finite element space can be taken for $\pi^H(\Omega)$.

$$\mathbf{L}^H \subset \mathbf{L}^h \subset \mathbf{L}$$

Example: for μ=h or H

$$\mathbf{X}^\mu := \{\mathcal{C}^0 \text{ piecewise linear (vectors) on } \pi^\mu(\Omega)\},$$
$$\mathbf{L}^\mu := \{L^2 \text{ discontinuous, piecewise constant (symmetric tensors) on } \pi^\mu(\Omega)\}$$

Note that $L^\mu = \nabla^s X^\mu$.

The idea of the method is to add global eddy viscosity to the FEM and to subtract its effects on the large scales as follows:

Find $\mathbf{u}^h : [0,T] \to \mathbf{X}^h$, $p^h : (0,T] \to Q^h$, and $\mathbf{g}^H : (0,T] \to \mathbf{L}^H$ satisfying

$$\begin{cases} (\mathbf{u}_t^h, \mathbf{v}^h) + a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) - (p^h, \nabla \cdot \mathbf{u}^h) + (q^h, \nabla \cdot \mathbf{u}^h) \\ \quad + (\nu_T \nabla^s \mathbf{u}^h, \nabla^s \mathbf{v}^h) - (\nu_T \mathbf{g}^H, \nabla^s \mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h), \ \forall \mathbf{v}^h \in \mathbf{X}^h, \forall q^h \in Q^h, \\ \qquad (\mathbf{g}^H - \nabla^s \mathbf{u}^h, \ell^H) = 0, \quad \forall \ell^H \in \mathbf{L}^H \end{cases}$$

$$\mathbf{g}^H = P_H(\nabla^s \mathbf{u}^h), \quad \text{where } P_H' : \mathbf{L} \to \mathbf{L}^H \text{ is the } L^2 \text{ orthogonal projector.}$$

Last term on rhs in 1st equation can be written as: $(\nu_T [(\nabla^s \mathbf{u}^h) - P_H(\nabla^s \mathbf{u}^h)], \nabla^s \mathbf{v}^h)$

Variational Multi-scale Model

With $P_H : \mathbf{L} \to \mathbf{L}^H$ *the* $L^2(\Omega)$ *orthogonal projector, define*

$$\overline{(\nabla^s \mathbf{u}^h)} := P_H(\nabla^s \mathbf{u}^h), \qquad (\nabla^s \mathbf{u}^h)' = (\mathbb{I} - P_H)(\nabla^s \mathbf{u}^h).$$

Then this term can be simply written as: $(\nu_T(\nabla^s \mathbf{u}^h)', (\nabla^s \mathbf{u}^h)')$.

Theorem *The method is equivalent to: find* $\mathbf{u}^h : [0,T] \to \mathbf{X}^h$*, and* $p^h : (0,T] \to Q^h$ *satisfying*

$$(\mathbf{u}_t^h, \mathbf{v}^h) + a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) - (p^h, \nabla \cdot \mathbf{v}^h) + (q^h, \nabla \cdot \mathbf{u}^h)$$
$$(\nu_T(\nabla^s \mathbf{u}^h)', (\nabla^s \mathbf{v}^h)') = (\mathbf{f}, \mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{X}^h,\ q^h \in Q^h.$$

The following theorem assures stability of this method:

Theorem *The solution* $\mathbf{u}^h$ *of satisfies* $\forall t \in (0,T]$*:*

$$\frac{1}{2}\|\mathbf{u}^h(\cdot,t)\|^2 + \int_0^t \left[\frac{2}{Re} \|\nabla^s \mathbf{u}^h\|^2 + \int_\Omega \nu_T |(\nabla^s \mathbf{u}^h)'|^2 \, d\mathbf{x} \right] dt'$$
$$= \frac{1}{2}\|\mathbf{u}^h(\cdot,0)\|^2 + \int_0^t (\mathbf{f}, \mathbf{v}^h)(t')\, dt'.$$

Multiscale decomposition of the deformation induces a multiscale decomposition of the velocities

Variational Multi-scale Model

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : (q, \nabla \cdot \mathbf{v}) = 0 \quad \forall\, q \in Q\},$$
$$\mathbf{V}^{\mu} := \{\mathbf{v}^{\mu} \in \mathbf{X}^{\mu} : (q^{\mu}, \nabla \cdot \mathbf{v}^{\mu}) = 0 \quad \forall\, q^{\mu} \in Q^{\mu}\}.$$

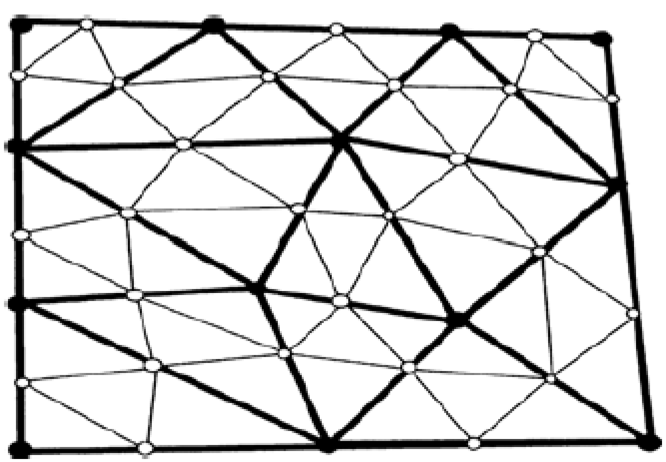
Definition (Elliptic projection). *For* $\mu = h, H$, $P_E^{\mu} : \mathbf{X} \to \mathbf{V}^{\mu}$ *is the projection operator satisfying*

$$(\nabla^s[\mathbf{w} - P_E(\mathbf{w})], \nabla^s \mathbf{v}^{\mu}) = 0, \quad \forall\, \mathbf{v}^{\mu} \in \mathbf{V}^{\mu}.$$

Theorem. Multiscale deformation decomposition is VMM with:

$$\mathbf{u}^h = \overline{\mathbf{u}} + (\mathbf{u}^h)', \quad \overline{\mathbf{u}} := P_E \mathbf{u}^h \in \mathbf{X}^H, \qquad (\mathbf{u}^h)' = (\mathbb{I} - P_E^H)\mathbf{u}^h.$$
$$(\nu_T \nabla^s (\mathbf{u}^h)', \nabla^s (\mathbf{v}^h)') = (\nu_T \nabla^s \mathbf{u}^h, \nabla^s \mathbf{v}^h) - (\nu_T P_H(\nabla^s \mathbf{u}^h), P_H(\nabla^s \mathbf{v}^h)).$$

$$\mathbf{X}^{\mu} := \{\mathcal{C}^0 \text{ piecewise linear on } \pi^{\mu}(\Omega)\}. \quad \overline{\mathbf{X}} = \mathbf{X}^H = \operatorname{span}\{\phi_N(\mathbf{x}) : \text{ all vertices } N \in \pi^H(\Omega)\}^3.$$



$$\mathbf{X}_b' := \operatorname{span}\{\phi_N(\mathbf{x}) : \text{ all vertices } N \in \pi^h(\Omega),\ N \notin \pi^H(\Omega)\}.$$

φ_N(x) is the usual piecewise linear finite element basis function associated with vertices of $\pi^H(\Omega)$

Ligth nodes correspond to velocity fluctuations

Conclusions

- VMM is LES model and can be implemented as finite element method
- Stability can be assured by typical conditions (LBB)
- Due to the fact that the VMM is derived from equivalent formulation of NS equations an approximate solution can be treated as approximation of the DNS problem ($u^h = \bar{u} + (u^h)'$)
- Because coarse mesh is used computation time can be decreased
- Usage of bubble functions results in easier to solve discrete equations

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With 32 Figures

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